CONTINUOUS QUOTIENTS FOR LATTICE ACTIONS ON COMPACT SPACES

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Abstract. Let $\Gamma < SL_n(\mathbb{Z})$ be a subgroup of finite index, where $n \geq 5$. Suppose Γ acts continuously on a manifold M, where $\pi_1(M) = \mathbb{Z}^n$, preserving a measure that is positive on open sets. Further assume that the induced Γ action on $H^1(M)$ is non-trivial. We show there exists a finite index subgroup $\Gamma' < \Gamma$ and a Γ' equivariant continuous map $\psi : M \to \mathbb{T}^n$ that induces an isomorphism on fundamental group.

We prove more general results providing continuous quotients in cases where $\pi_1(M)$ surjects onto a finitely generated torsion free nilpotent group. We also give some new examples of manifolds with Γ actions.

1. Introduction

Let G be a semisimple Lie group with $\mathbb{R}$-rank$(G) \geq 2$, and $\Gamma < G$ a lattice. In this paper we seek to examine the relationship between the topology and dynamics of measure preserving actions of Γ on a compact manifold M. More precisely, we seek relations between the fundamental group of M and the structure of both M and the Γ action on M. The simplest version of our main theorem is:

Corollary 1.1. *Let $\Gamma < SL_n(Z), n \geq 3$ be a subgroup of finite index. Suppose Γ acts on a compact manifold M preserving a measure that is positive on open sets. Assume $\rho : \pi_1(M) \to \mathbb{Z}^n$ is Γ equivariant, that the action of Γ on $\mathbb{Z}^n$ is given by the standard representation of $SL_n(\mathbb{R})$, and that the Γ action lifts to* $\ker \rho$. *(The lifting condition is automatic provided $n \geq 5$). Then there is a finite index subgroup $\Gamma' < \Gamma$ and a Γ' equivariant map $\psi : M \to \mathbb{T}^n$ which induces the map ρ on fundamental groups.*

The actual result applies more generally to certain kinds of actions (described precisely below) of lattices on manifolds whose fundamental group surjects onto a torsion free finitely generated nilpotent group. In all cases we produce a continuous map from our action to an algebraically defined action on a nilmanifold. In particular, the theorem applies to the surgery examples of Katok and Lewis, and the quotient recaptures the torus action on which the surgeries were performed [KL]. For some of the examples with more complicated fundamental group it will be necessary to pass to a finite cover before our theorem applies. For all of these examples, the map to the torus simply collapses certain invariant submanifolds whose fundamental groups

map to 0 in $\mathbb{Z}^n$. In section 4, we prove a purely topological result that shows that this is essentially what all such maps must look like.

This work is related to work of the first author, who shows in [F] that under different hypothesis there are measurable maps to the same standard examples. The argument of [F] also shows that fundamental groups of manifolds with measure preserving Γ actions are of arithmetic type under fairly mild hypotheses relating the dynamics to the topology, unless the fundamental group admits no linear representation. Further work of the first author with Zimmer [FZ] produces linear representations of the fundamental group under stronger geometric assumptions. Combining these results gives that either the action on fundamental group is trivial or that the fundamental group contains a direct factor that is finitely generated and nilpotent on which Γ acts nontrivially. These results give some evidence that our assumptions are not atypical, at least in the case that $\Gamma \rightarrow (\mathrm{Out}(\pi_1(M)))$ is nontrivial.

We also discuss several examples that show that for our general result one cannot expect to improve the regularity of the quotients without stronger assumptions. Note that our theorems apply not just to manifolds but to any topological spaces satisfying standard covering space theory, i.e. any connected, locally path-connected, semi-locally 1-connected, locally compact, separable, metrizable space with finitely generated fundamental group.

2. **Preliminaries**

We assemble here some basic facts from topology and dynamics that will be used in the proof.

Suppose a finitely generated group Γ acts continuously on a manifold M. Let $\tilde{M}$ be any cover of M and D and Δ be the deck group of $\tilde{M}$ over M and the group of lifts of the Γ action, respectively. There is an exact sequence:

$$1 \longrightarrow D \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow 1$$

and the Γ action lifting to $\tilde{M}$ is equivalent to this exact sequence splitting. In particular, elementary group cohomology shows that the sequence splits if $H^2(\Gamma, D) = 0$ and the map $\Gamma \rightarrow \mathrm{Out}(D)$ given by the sequence lifts to a map $\Gamma \rightarrow \mathrm{Aut}(D)$.

Suppose Γ, a group, acts on a measure space M preserving a finite measure. A cocycle is a measurable map $\alpha : \Gamma \times M \rightarrow H$ where H is a group and α satisfies the equation $\alpha(g_1 g_2, m) = \alpha(g_1, g_2 m)\alpha(g_2, m)$ for all $g_1, g_2 \in \Gamma$ and all $m \in M$. Two cocycles α and β are cohomologous if there is a measurable map $\phi : M \rightarrow H$ such that $\alpha(g, m) = \phi(gm)^{-1}\beta(g, m)\phi(m)$ for all $g \in \Gamma$ and almost all $m \in M$. Now assume that $\Gamma < G$ is a lattice where G is a semisimple Lie group, that the Γ action on M is ergodic, and that H is an algebraic group. By results of Zimmer, for any cocycle $\alpha : \Gamma \times M \rightarrow H$, there is a minimal subgroup $L < H$, unique up to conjugacy, such that α is cohomologous to a cocycle taking values in L. The group L is called the algebraic hull of the cocycle [Z1].

We will also need the following definition:

Definition 2.1. *A representation $\sigma : \Gamma \to GL_n(\mathbb{R})$ is weakly hyperbolic if there is no invariant subspace $V < \mathbb{R}^n$ where all the eigenvalues of all elements of Γ have modulus one.*

3. **Main Theorem and Proof**

Suppose Γ, a finitely generated group, acts on a compact manifold M and there is a surjection $\rho : \pi_1(M) \to \Lambda$, where Λ is a finitely generated torsion free nilpotent group.

If the action of Γ lifts to the cover of M corresponding to $\ker \rho$, then we have a map $\sigma : \Gamma \to \mathrm{Aut}(\Lambda)$. By a theorem of Malcev, this gives a map $\sigma : \Gamma \to \mathrm{Aut}(N)$ where N is a simply connected nilpotent Lie group, containing Γ as a lattice. We can view σ as a representation $\sigma : \Gamma \to GL(\mathfrak{n})$ where $\mathfrak{n} = \mathrm{Lie}(N)$.

Definition 3.1. *Under the conditions described above, we say that the Γ action on M is π_1-hyperbolic if $\sigma : \Gamma \to GL(\mathfrak{n})$ is weakly hyperbolic.*

Note that our definition of π_1 hyperbolic includes the assumption that the Γ action lifts to the appropriate cover of M.

Theorem 3.2. *Let $\Gamma < G$ be an irreducible lattice, where G is a semisimple Lie group with all simple factors of $\mathbb{R}$-rank$(G) \geq 2$. Suppose that Γ acts on a compact manifold M preserving a measure that is positive on open sets. Assume there is a surjection $\rho : \pi_1(M) \to \Lambda$ where Λ is a finitely generated torsion free nilpotent group and that the Γ action is π_1-hyperbolic. Then there is a finite index subgroup $\Gamma' < \Gamma$ and a Γ' equivariant map $\psi : M \to N/\Lambda$ where the action on N/Λ is defined by extending the action of Γ' on Λ to N. Furthermore, the map ψ_* on fundamental group is equal to ρ above.*

In the case of M having abelian fundamental group, this gives corollary 1.1 from the introduction. We need only explain the remark that the lifting of the action is automatic in the case where $n \geq 5$. To see this, we need only see that $H^2(\Gamma, \mathbb{Z}^n) = 0$, since $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{Out}(\mathbb{Z}^n)$. By theorem 4.4 of [B], we know that $H^2(\Gamma, \mathbb{R}^n) = 0$. By looking at the long exact sequence in cohomology corresponding to $1 \to \mathbb{Z}^n \to \mathbb{R}^n \to \mathbb{T}^n \to 1$, this implies that $H^2(\Gamma, \mathbb{Z}^n) = H^1(\Gamma, \mathbb{T}^n)$ which is finite and vanishes if we pass to a subgroup of finite index. This allows us to lift the action of this subgroup of finite index, which is sufficient for our purposes. Note that the statement in the abstract follows from the corollary, since Margulis' superrigidity theorem and the action on $H^1(M)$ being nontrivial imply that the action on $\mathbb{Z}^n$ is indeed given by the standard representation.

Proof. Since $\rho : \pi_1(M) \to \Lambda$ is a surjection, we have a continuous map $f : M \to N/\Lambda$. This follows since N is contractible and N/Λ is an Eilenberg-MacLane space for Λ and hence M has a continuous map to N/Λ inducing ρ on fundamental groups which is canonical up to homotopy.

We can lift f to a Λ equivariant map $\tilde{f} : \tilde{M} \to N$ where $\tilde{M}$ is the cover of M corresponding to $\ker \rho$.

We consider the map $\tilde{\alpha} : \Gamma \times \tilde{M} \to N$ defined by

$$\tilde{\alpha}(\gamma, m) = \tilde{f}(\gamma m)(\gamma f(m))^{-1}.$$

This is clearly a measure of the extent to which $\tilde{f}$ fails to be equivariant, and is defined since we have assumed the Γ action lifts to $\tilde{M}$. First we show that $\tilde{\alpha}$ descends to a map $\alpha : \Gamma \times M \to N$. To see this, let $m \in \tilde{M}$ and $m\lambda$ be any translate of m where $\lambda \in \Lambda$ is viewed as a deck transformation of $\tilde{M}$ over M. It suffices to show that $\alpha(\gamma, m) = \alpha(\gamma, m\lambda)$, but

$$\begin{aligned} \alpha(\gamma, m\lambda) &= \tilde{f}(\gamma m\lambda)(\gamma f(m\lambda))^{-1} \\ &= \tilde{f}((\gamma m)(\gamma\lambda))(\gamma(f(m)\lambda))^{-1} \\ &= \tilde{f}(\gamma m)(\gamma\lambda)((\gamma f(m))\gamma\lambda))^{-1} \\ &= \tilde{f}(\gamma m)(\gamma\lambda)(\gamma\lambda)^{-1}(\gamma f(m))^{-1} \\ &= \tilde{f}(\gamma m)(\gamma f(m))^{-1} \end{aligned}$$

since $\tilde{f}$ is Λ equivariant and the action of Γ on Λ induced by the action on $\pi_1(M)$ is the same as the action of Γ on $\Lambda < N$.

We now look at the map $\beta : \Gamma \times M \to \Gamma \ltimes N$ defined by $\beta(\gamma, m) = (\gamma, \alpha(\gamma, m))$. A simple computation verifies that β is a cocycle over the Γ action on M. We can view β as a cocycle into $G \ltimes N$ by the natural inclusion. For now, we assume the action is ergodic. By results of Lewis and Zimmer, the algebraic hull, L, of this cocycle will be reductive with compact center. Let $L_0 < L$ be the connected component of the identity in L. By passing to a finite ergodic extension of the action on $X = M \times L/L_0$ we have a cocycle $\beta : \Gamma \times X \to G \ltimes N$ (still called β) with algebraic hull L_0. Note that $\beta(m, l)$ depends only on m. Since any connected reductive subgroup of $G \ltimes N$ is conjugate to a subgroup of G, we can assume that $L_0 < G$. This means that the cocycle on all of X is cohomologous to one taking values in G, in other words $\beta(\gamma, x) = \phi(\gamma x)^{-1}\delta(\gamma, x)\phi(x)$ where $\phi : X \to G \ltimes N$ is a measurable map and $\delta : \Gamma \times X \to G \ltimes N$ is a cocycle taking values entirely in G. Write $\phi(x) = (\phi_1(x), \phi_2(x)) = (\phi_1(x), 1_N)(1_G, \phi_2(x))$. Since $\delta = (\delta_1, 1_N)$, computing the cocycle equivalence above in components yields that $\beta(\gamma, x) = (1_G, \phi_2(\gamma x)^{-1}(\gamma, 1_N)(1_G, \phi_2(x))$.

We now show that ϕ_2 is continuous. The argument follows exactly as in Lemma 6.5 of [MQ]. For the reader's convenience we repeat here the case where $N = \mathbb{R}^n$. Essentially the idea is to use the fact that $\mathbb{R}^n$ is spanned by contracting directions for elements of Γ and to show that along any contracting direction, ϕ_2 can as the limit of iterated contractions of α.

For $\gamma \in \Gamma$ let $E(\gamma)$ and $F(\gamma)$ be subspaces of $\mathbb{R}^n$ that are the generalized eigenspaces of γ with eigenvalues of absolute value > 1 and ≤ 1 respectively. Clearly $\mathbb{R}^n = E(\gamma) \oplus F(\gamma)$, and the assumption of weak hyperbolicity implies

that $\mathbb{R}^n$ is spanned by $\{E(\gamma)|\gamma\in\Gamma\}$. We show continuity of ϕ_2 by showing continuity of ϕ_2 projected onto any $E(\gamma)$. For any function $h : M\to\mathbb{R}^n$ we write $h_{E(\gamma)}$ for the composition of the function with projection on $E(\gamma)$.

Looking at what the cocycle condition on β implies for α we see that $\phi_2(x) = \gamma^{-1}\phi_2(\gamma x) + \gamma\alpha(\gamma, m)$ where $x = (m, f)\in X$. Iterating this equality and projecting to $E(\gamma)$ gives

$$\phi_{2E(\gamma)}(x) = \sum_{i=1}^{n}(\gamma^i)^{-1}|_{E(\gamma)}\alpha_{E(\gamma)}(\gamma, \gamma^{i-1}m) + \gamma^{n-1}|_{E(\gamma)}\phi_{2E(\gamma)}(\gamma^n x).$$

Since the eigenvalues of $\gamma^{-1}|_{E(\gamma)}$ all have absolute value < 1, on a set of full measure $\gamma^{n-1}|_{E(\gamma)}\phi_{2E(\gamma)}(\gamma^n x)\to 0$ as $n\to\infty$. So we have:

$$(*) \qquad \phi_{2E(\gamma)}(x) = \sum_{i=1}^{\infty}(\gamma^i)^{-1}|_{E(\gamma)}\alpha_{E(\gamma)}(\gamma, \gamma^{i-1}m)$$

which converges uniformly since $\alpha(\gamma, -)$ is continuous and bounded function on M. This shows both that ϕ_2 is continuous and is a function on M that is independent of the finite ergodic extension X.

If the action is not ergodic, we simply carry out the analysis above on each ergodic component. We will get a function ϕ_2 as above for each component. Since $(*)$ above shows how to compute ϕ_2 explicitly on any ergodic component, we see that ϕ_2 is a well defined continuous function from X to $\mathbb{R}^n$. From $(*)$ it is clear that ϕ_2 descends to a continuous function M to $\mathbb{R}^n$. When $\mathbb{R}^n$ is replace by a more general simply connected nilpotent group, the analysis becomes more complicated but follows exactly as in [MQ].

Now $\beta(\gamma, m) = (\gamma, \tilde{f}(\gamma m)(\gamma\tilde{f}(m))^{-1})$ where we are actually choosing some lift of the point m to $\tilde{M}$. Substituting this in above and computing the N factor gives $\tilde{f}(\gamma m)(\gamma\tilde{f}(m))^{-1} = \phi_2(\gamma m)^{-1}(\phi_2(m))$. We can lift ϕ_2 to a map from $\tilde{M}$ to N, and then rearranging the last expression gives $\tilde{\phi_2}(\gamma m)\tilde{f}(\gamma m) = \gamma\tilde{\phi_2}(m)\gamma\tilde{f}(m) = \gamma(\tilde{\phi_2}\tilde{f})$. This shows that the map $(\tilde{\phi_2})(\tilde{f}) : \tilde{M}\to N$ is Γ equivariant. Since $\tilde{\phi_2}$ is Λ invariant, $\tilde{f}$ is Λ equivariant and Λ acts on the right on $\tilde{M}$, we see that the map $\phi_2 f : M\to N/\Lambda$ is also Γ equivariant. Note that $\phi_2 : M\to N$ is a map into a contractible space and so isotopically trivial. This implies that $\phi_2 f$ is in the same isotopy class as f and therefore that $\phi_2 f$ induces the map ρ on fundamental group.

□

4. **Examples**

For the sake of clarity we discuss the case of Corollary 1.1 only, although much can easily be generalized to the general case of Theorem 3.2. Throughout Γ will refer to a finite index subgroup of $SL_n\mathbb{Z}$.

We start with some non-trivial examples to which our theorem applies. These examples of exotic $SL_n(\mathbb{Z})$ actions are due to Katok and Lewis ([KL]). They are constructed from the standard action on $\mathbb{T}^n$ by blowing up the

fixed point, so that it becomes a copy of $\mathbb{R}P^{n-1}$ with the standard action of $SL_n\mathbb{Z}$. What is not at all obvious that the resulting manifold has an invariant analytic structure and volume form. Indeed, in order to make the action preserve a volume form one must use a differential structure which is not the obvious one. The continuous map to the torus, collapsing the $\mathbb{R}P^{n-1}$, is not smooth with respect to this new smooth structure. It is immediate from the proof of Theorem 3.2 that the map is unique, since it is given explicitly by (*) in terms of the action. This shows that the regularity of the semi-conjugacy in Theorem 3.2 cannot be improved, even when the action is analytic.

There are further examples of exotic actions of $SL_n(\mathbb{Z})$, which were originally constructed by Weinberger. Take $\mathbb{T}^n$ and remove some finite invariant set. The resulting manifold can be compactified to a manifold with boundary by adding the spheres in the tangent bundle at each point. The action of Γ on the boundary is the standard linear actions. These actions extend over n-balls - just think of the ball as $\mathbb{R}^n$ with the linear Γ action as the action on the sphere at infinity. Gluing in these balls gives a closed manifold with a Γ action. The underlying manifold is still the torus, but the action is different. There is no invariant measure on the whole torus, but there is on the complement of the disks. Thus our theorem says there is a continuous map from this complement to the torus. Indeed the map which simply collapses the balls we glued in back to points is continuous and equivariant.

It is possible to combine Weinberger's construction with some surgery to produce a new class of examples. Take $(X, \partial X)$ any compact manifold with boundary, let N be Weinberger's example cross a ∂X. Inside of N is a ball cross ∂X. Remove the interior of this, and glue in X cross an $n-1$ sphere in its place to get M. If $\pi_1(X)$ is trivial, $\pi_1(M) = \mathbb{Z}^n$. As a specific example, taking X to be the m ball gives an example of a Γ-manifold with $\pi_1 = \mathbb{Z}^n$ but which is definitely not, even non-equivariantly, a bundle over $\mathbb{T}^n$.

In the above examples the map to the torus is always well behaved off of a submanifold which is collapsed. The following lemma, in some ways a converse to the main theorem, shows that this collapsing must occur in general.

Lemma 4.1. *Let Z be any compact, connected, space with a Γ action. Any non-constant equivariant map to the torus surjects $\pi_1(Z)$ onto a finite index subgroup of $\mathbb{Z}^n$.*

Proof. Suppose not. Then, since the image of $\pi_1(Z)$ is a Γ invariant infinite index subgroup of $\mathbb{Z}^n$, it must be trivial. Thus we aim to show that any null homotopic map is constant. Let $f : Z \to \mathbb{T}^n$ be a null homotopic equivariant map.

The image of f in $\mathbb{T}^n$ is closed, invariant, and connected. Hence, since we assume the map non-constant, it must be surjective. In particular, some z_0 in M maps to 0 in $\mathbb{T}^n$. Lift f to a map $F : Z \to \mathbb{R}^n$ such that $F(z_0) = 0$.

Fix a $\gamma \in \Gamma$. Since F covers an equivariant map, we know that

$$\tau_\gamma(z) = F(\gamma z) - \gamma F(z)$$

takes values in $\mathbb{Z}^n$. Since it is continuous in z, this implies it is constant in z. We define τ_γ to be the common value. It is easy to see that $\tau : \Gamma \to \mathbb{Z}^n$ is a 1-cocycle, in other words

$$\tau_{\gamma\sigma} = \tau_\gamma + \gamma\tau_\sigma.$$

We can evaluate τ_γ at z_0, which yields $\tau_\gamma = F(\gamma z_0)$. Since the image of F is compact, this shows that τ_γ is bounded indepependant of γ. The cocycle identity then shows that $\gamma\tau_\sigma$ is bounded independantly of γ and σ. In particular, τ_σ has bounded Γ orbit and so we have $\tau_\sigma = 0$ for all σ. This means precisely that F is equivariant. This finishes the proof as $F(Z)$ is then a bounded, invariant set in $\mathbb{R}^n$, and thus is $\{0\}$.

□

In all of these examples, the map to the torus is nice everywhere except the pre-image of a finite invariant set. We believe that will always be the case. Note that if the map had any regularity, then by Sard's theorem the critical values would be measure zero. Since they are closed and invariant this would limit them to a finite invariant set, and thus we would know that off this finite set our manifold is a bundle over the torus. We know from the Katok-Lewis examples that the map need not be C^1, even for analytic actions. The map in that case is, however, still analytic off a lower dimensional submanifold.

Question 4.2. *If, in the statement of the main theorem, one assumes the action to have some regularity, is the map also regular away from the pre-image of a finite invariant set in the torus?*

Even if regularity does not hold, one can still hope the map must be "taut" in some sense. The last lemma is one example of the sort of substitute for regularity. At the very least one would like to be able to rule out space filling curves in this context, so as to prove:

Conjecture 4.3. *Any compact manifold with $\pi_1 = \mathbb{Z}^n$ and a Γ action which induces the standard action on π_1 must be of dimension at least n.*

Here we have no assumption of an invariant measure. In all the examples there is an invariant measure, at least on a large open set. Is this always the case?

Question 4.4. *If $SL_n(\mathbb{Z})$ acts on a compact manifold with fundamental group $\mathbb{Z}^n$, inducing the standard action on π_1, is there always an invariant measure? Is there always an invariant measure whose support contains an open set?*

One is tempted to view the torus with the standard action as some kind of equivariant classifying space for Γ actions with $\pi_1 = \mathbb{Z}^n$, and to view our

theorem as proving the existence of a classifying map One cannot hope for general results of this type - the use of superrigidity is not just an artifact of the method. Even when there is a clear candidate classifying space, and the action there is hyperbolic, the analog of our theorem need not hold.

Consider the action of $SL_2(\mathbb{Z})$ on $\mathbb{T}^2$. Since SL_2 acts hyperbolically on the torus, one might expect our theorem to cover this case. It does not: take Γ any torsion free subgroup of finite index in $SL_2(\mathbb{Z})$. Such a Γ is free. Starting with the standard action of Γ on $\mathbb{T}^2$, conjugate the action of one of the free generators by a homeomorphism homotopic to the identity, and leave the action of the remaining generators unchanged. Since Γ is free, this still generates an action of Γ. There is no equivariant map of this torus to the standard one. This follows from the uniqueness of the map conjugating a single Anosov homeomorphism to a linear Anosov automorphism, which follows from the same reasoning that shows the map in Theorem 1.1 is unique (or see [KH]).

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